Random walk with an exponentially varying step

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A random walk with exponentially varying step, modeling damped or amplified diffusion, is studied. Each step is equal to the previous one multiplied by a *step factor* s $(0<s<\infty)$. There is a symmetry under the transformation $s\rightarrow 1/s$ relating different processes. For $s<1/2$ and $s>2$, the process is retrodictive (i.e., every final position can be reached by a unique path) and the set of all possible final points after infinite steps is fractal. For step factors in the interval $[1/2,2]$, some cases result in smooth density distributions, other cases present overlapping self-similarity and there are values of the step factor for which the distribution is singular without a density function.

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I. INTRODUCTION AND DEFINITION OF THE MODEL

The standard random walk, with a constant step, is a powerful tool for studying several physical processes such as diffusion, transport [1], aggregation, structure formation [2–4], and diffusion controlled reactions [5–7]. In this paper, we will study a random walk with a step length varying exponentially in time. This problem was suggested in the study of the effect of state reduction (collapse) in multiple position or momentum observations of a quantum-mechanical particle [8]. The detection of such a particle in a one-dimensional space, at the right or at the left of its position expectation value, would leave it in a state, shifted to the right or to the left, with a reduced position indeterminacy. Every subsequent observation of the same type would correspond to a right or left jump of reduced length. Under some assumptions, the step length will vary exponentially. However, beyond the original motivations for the process, the problem posed turned out to be interesting in its own right.

Let us consider a random walk starting at position $X=0$ in a one-dimensional space with jumps to the right or to the left with equal probability. In the first step, the particle jumps a distance ℓ and in the following step, the distance jumped is $s\ell$, where the *step factor* s can be any positive real number. So, the Nth step is of length ℓs^{N-1} to the right or to the left with equal probability. (We will always take $\ell=1$ except at one comment later where the limit $\ell\rightarrow 0$ is considered.) If $s<1$ $(s>1)$, we have exponentially decreasing (increasing) steps and if $s=1$ we recover the standard random walk. Let the set $\{X_I(N,s)\}$ for $I=0,1,\ldots,2^N-1$ denote all possible end positions of the particle after N steps with step factor s. These values are

$$X_I(N,s)=\sum_{k=0}^{N-1}\sigma_{I,k}s^k, \tag{1}$$

where $\sigma_{I,k}=\pm 1$ is a matrix whose rows are all possible combinations of N plus or minus signs. (How the rows are ordered by the index I is for the moment not important but we will later choose a particular ordering.) When no confusion can arise, we will suppress the arguments N and/or s. This set has a lower and upper bound corresponding to the cases where all signs are negative and positive. Then,

$$-\frac{1-s^N}{1-s}\leqslant X_I\leqslant\frac{1-s^N}{1-s}. \tag{2}$$

We will call this process *centered*. It is convenient to define an equivalent *drifting* process where, on the Nth step, there is equal probability of staying at the same place or jumping to the right a distance of length s^{N-1}. Let us denote by $\{Y_I(N,s)\}$, $I=0,1,\ldots,2^N-1$ the set of all possible positions after N steps. We have then

$$Y_I(N,s)=\sum_{k=0}^{N-1}b_{I,k}s^k, \tag{3}$$

where $b_{I,k}=(\sigma_{I,k}+1)/2=0$ or 1. Therefore,

$$\{Y_I\}=\left\{\frac{1}{2}X_I+\frac{1}{2}\frac{1-s^N}{1-s}\right\}. \tag{4}$$

Clearly, this process is identical to the previous one but shifted to the right such as to make the lower bound equal to zero but with the same upper bound as in Eq. (2). Another process, equivalent to the centered one, that allows more physical insight, is obtained by considering a particle moving in one dimension, starting from $X(0)=0$ at $t=t_0$ with constant speed $V=\pm 1$. At the times $t_0=0,t_1,t_2,\ldots,t_{N-1}$ it is decided, with equal probability, to keep moving in the same direction or to change direction. The position of the particle at time t_N will be then given by

$$X(t_N)=\pm(t_1-t_0)\pm(t_2-t_1)\pm\cdots\pm(t_N-t_{N-1}). \tag{5}$$

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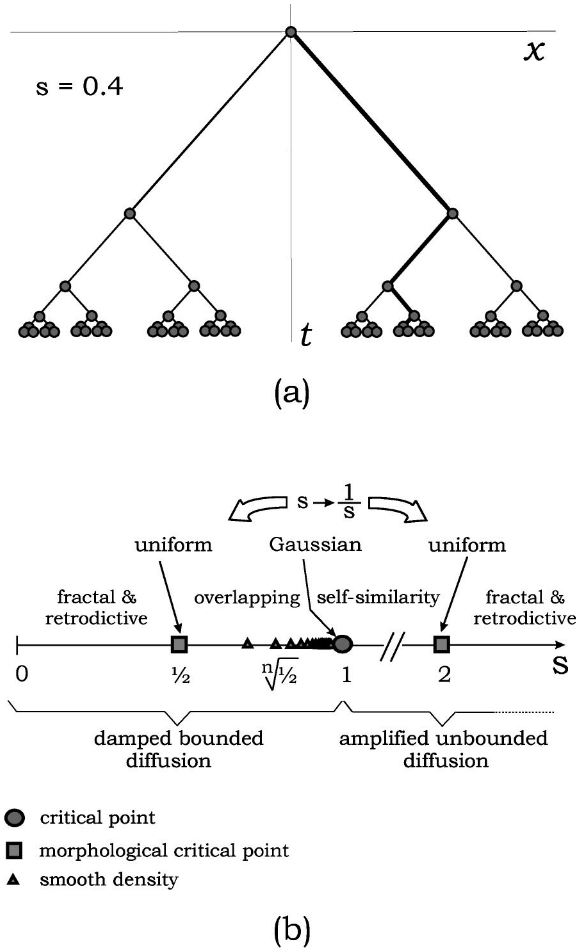

FIG. 1. (a) Space-time evolution of the process for $s=0.4$. For this value of the step factor the paths never cross each other. The largest gap is at the center and smaller gaps are created at all levels. (b) Summary of the dependence of the process on the step factor.

Now if $(t_k-t_{k-1})=\tau$ is a constant, we have the standard random walk. If $(t_k-t_{k-1})=\tau_k$ is taken from some distribution, we obtain the standard random walk with random varying step [9] and if $(t_k-t_{k-1})=s^{k-1}$ we get the process defined above. A space-time diagram for this process is shown in Fig. 1(a) for $s=0.4$. In these models we can assume that the change in velocity can be done with an energy exchange with the environment and, as a consequence, different paths will have different energy cost and the concept of temperature could be introduced. This model is related to the Ising one-dimensional model and we will not treat it further in this paper. The space-time representations of the process, as in Fig. 1(a), are useful in order to understand some of its features. For instance, the formation of holes when $s<1/2$ is understood because the two branches corresponding to the first step being to the right or to the left do not overlap, creating a central region that cannot be reached by any path. If $1/2<s<1$, the two branches will overlap and clustering regions will be generated. More on this will be said later.

II. GENERAL FEATURES

In the drifting picture, it is easy to relate the set of final positions after N and $N+1$ steps. After N steps, we can have two cases: either $b_{I,N}=0$ and the set after N steps is kept, or $b_{I,N}=1$ and we generate a new set obtained from the previous one by adding s^N to all its elements. We have therefore

$$\{Y_I(N+1)\}=\{Y_I(N)\}\cup\{Y_I(N)+s^N\}. \tag{6}$$

The corresponding relation for the centered picture is

$$\{X_I(N+1)\}=\{X_I(N)\pm s^N\}. \tag{7}$$

It is convenient to define, for the drifting case, the *gap* G_{N+1} as the difference between the smallest element of the second set and the largest one of the first set in the right-hand side of Eq. (6) above:

$$G_{N+1}=s^N-\frac{s^N-1}{s-1}. \tag{8}$$

For step factors $s\geqslant 2$, the gap is positive for all N and self-similarity patterns are produced because, from one step to the next, the same set of points is created, but shifted to the right by an amount large enough to keep the two sets separated. One can also check that, in the case $s\geqslant 2$, the gap is nondecreasing, that is, $G_{N+1}\geqslant G_N$. Considering the case $N=1,2,\ldots$ we infer that, when $s\geqslant 2$, the gap G_N is the largest spacing in the set of final positions after N steps and, for large N, it diverges like s^{N-1}. For each integer number of steps, there is some value of the step factor in the interval $s\in[1,2]$, the root of $s^N(s-2)+1=0$, such that the corresponding gap G_{N+1} vanishes. For values of the step factor smaller than the root of $s^N(s-2)+1=0$, the gap is negative and the two sets in Eq. (6) overlap destroying the *strict* self-similarity of the set $\{Y_I\}$. However in most cases, the overlap does not wipe out completely the repetitive feature of self-similarity. Only for some special cases, that we will study later, the self-similarity features are completely wiped out. We may denote by *overlapping self-similarity* the remaining self-similarity that the overlap did not erase. There are values of s (less than 2) and N such that the two overlapping sets in the right-hand side of Eq. (6) will contain the same value. This means that there may be different paths reaching the same final position. More on this will be said later.

Dividing Eq. (1) or (3) by s^{N-1}, we find that there is a symmetry $s\to 1/s$ in the process. That is,

$$\left\{\frac{X_I(N,s)}{s^{N-1}}\right\}=\left\{X_{I'}\left(N,\frac{1}{s}\right)\right\}. \tag{9}$$

(The ordering I and I' in these sets is not the same.) In the process, the time T is represented by the number of steps N. Since the step length varies exponentially with N (time), the transformation $s\to 1/s$ is equivalent to a time reversal transformation $T\to -T$. The process is therefore invariant under the combined transformation $s\to 1/s, T\to -T$. This symmetry is useful because everything that we can prove for $s>1$ can be used in order to reach a corresponding conclusion for $s<1$ and vice versa. As an example of this we can prove that, for $s=2$ and $s=1/2$, the values in the set are uniformly distributed. Let us take $s=2$ in Eq. (3):

$$Y_I=\sum_{k=0}^{N-1} b_{I,k}2^k, \quad b_{I,k}=0 \text{ or } 1. \tag{10}$$

This is the expansion of all integers I in base 2 because we can choose the ordering of the rows of the matrix $b_{I,k}$ such that $Y_I=I$. Therefore the set $\{Y_I(N,2)\}$ contains all integers $0,1,2 \ldots ,2^N-1$ uniformly distributed. By the symmetry $s \to 1/s$, we conclude that the set $\{Y_I(N,1/2)\}$ is also uniformly distributed with 2^N values between 0 and $(2^N-1)/2^{N-1}$. The uniform distribution for $s=1/2$ is clearly illustrated by the construction of the corresponding space-time diagram as in Fig. 1(a).

From Eqs. (1) and (3) defining the process, one can easily conclude that every position reached after $m<N$ steps can be considered as the starting point of the process, but scaled by a factor s^m. In order to see this clearly, we just have to decompose the sum in two sums, the first one running up to s^{m-1}, and take common factor s^m from the second sum. This feature is also clear in the space-time diagrams where each point is the starting point of the same tree but with its size reduced (or enlarged if $s>1$). As a consequence of this, when $N\to\infty$, every pattern of the process will be repeated infinite times but each time scaled by an extra factor s. These repetitions generate the self-similarity (or overlapping self-similarity in the case $1/2<s<2$). In particular, if for some value of s, there is a path that returns to the origin after some number of steps, then every point can be also revisited after the same number of steps.

The first moment of the distribution of final positions in the centered picture is obviously zero. From Eq. (7) and considering that all end positions after N steps have equal probability $1/2^N$, we can easily derive a recurrence relation for the second moment,

$$\langle X^2\rangle_{N+1}=\langle X^2\rangle_N+s^{2N}, \tag{11}$$

that iterated results in

$$\sigma^2=\langle X^2\rangle_N=\sum_{k=0}^{N-1} s^{2k}=\frac{1-s^{2N}}{1-s^2}. \tag{12}$$

For $s<1$ the standard deviation is constant for large N, $\sigma^2 \to 1/(1-s^2)$. If $s>1$, it diverges like s^{2N} and for $s=1$ it grows like N, as is expected in the standard random walk. The step factor $s=1$ is therefore like a critical value that separates two different behaviors. On one side ($s<1$) we have *damped diffusion* within a bound domain and in the other side ($s>1$) we have *amplified diffusion* in an unbounded region. We will see later that the cases $s=1/2$ and $s=2$ are also critical values separating different behavior not apparent in Eq. (12), namely, fractal and nonfractal distribution of the end points of the process. These are like critical points separating two morphological phases. This dependence of the process on the step factor, as well as other features that will be explained later, are summarized in Fig. 1(b).

In general, all moments can be obtained from the moment generating function as

$$\langle X^r\rangle_N=\frac{1}{i^r}\frac{d^r\phi_N(u)}{du^r}\bigg|_{u=0}, \tag{13}$$

where

$$\phi_N(u)=\sum_{I=0}^{2^N-1}\frac{1}{2^N}\exp(iuX_I)=\prod_{k=0}^{N-1}\cos(us^k), \tag{14}$$

which implies the recurrence relation

$$\phi_{N+1}(u)=\cos(u)\phi_N(su). \tag{15}$$

III. FRACTAL, OVERLAPPING SELF-SIMILARITY AND SMOOTH DENSITIES

An interesting feature of the process for $s>2$, and therefore also for $s<1/2$ due to the symmetry $s\to 1/s$, is that it is *retrodictive* in the sense that a given end position can only be reached by one path, that is, by only one combination of the coefficients $\sigma_{I,k}$ or $b_{I,k}$. This is clearly seen in the drifting picture in the case $s>2$ because in this case Eq. (3) is the expansion of the number Y_I in the base s, and, of course, the ''digits'' $b_{I,k}$ are unique. [For $1/2<s<2$, Eq. (3) would no longer be the expansion of a number in a base.]

The last remark leads us to another interesting feature of the set of end positions in the case that $s<1/2$ or $s>2$ and when $N\to\infty$. In this case the sets are fractals because they correspond to the expansion of numbers in a base where *only* the digits 0 and 1 are taken and all others are excluded. The most famous example is when $s=1/3$ that results in the set of Cantor [2–4]. We can find the fractal dimension for $s>2$ to be $D=\ln 2/\ln s$ as follows: for fractals generated by growth processes, one can define the fractal dimension D by $M \sim L^D$, where M is the mass of the objects (the number of end points in our case) and L is its linear size [2–4,10]. In the drifting picture, and considering the cases $N\to N+1$, we see from Eq. (6) that $M\to 2M$ and from Eqs. (2) and (4) that $L\to L(1-s^{N+1})/(1-s^N)$. Combining these relations we obtain, for $N\to\infty$, the fractal dimension given above. Using the ''box counting'' method [2–4,10] or applying the transformation $s\to 1/s$ in the equation for D, we find the fractal dimension for $s<1/2$ to be $D=-\ln 2/\ln s$. From this fractal structure, it is clear that when s is outside the interval $[1/2,2]$ the end positions cannot be described by a continuous density function because there will be ''holes'' of all sizes unreachable with this random walk. We can see how these holes are built using the gap defined earlier. When $s>2$, the gap G_N grows like s^{N-1}. Even if we rescale the one-dimensional space dividing by the width of the distribution $\sigma\sim s^N$, we still have holes of finite size $\sim 1/s$. Furthermore we can use the symmetry $s\to 1/s$ and project the gap G_N divided by s^{N-1} as required by Eq. (9), and we obtain, in the drifting picture and when $N\to\infty$ and $s<1/2$, the size of the largest hole $(1-2s)/(1-s)$ that does not vanish when $N\to\infty$. Summarizing, for s outside the interval $[1/2,2]$ the process is retrodictive, it has a fractal structure and does not have a continuous limit.

The general analysis for the case when $s\in[1/2,2]$ is a very difficult problem. However, for some cases we have some results. Of course, the case $s=1$ is well-known with a binomial distribution for the end points that goes to a Gaussian when $N\to\infty$ and $\ell\to 0$ with constant $\sigma=\sqrt{N}\ell$. We have seen that in the cases $s=1/2$ and $s=2$ there are 2^N ending points uniformly distributed that in the limit $N\to\infty$ are represented by a uniform density function. In general, due to the symmetry $s\to 1/s$, we only need to solve the problem in the interval $(1/2,1)$.

If we try to solve the continuous problem when $N\to\infty$ in the same way as is done for the case $s=1$, we find that this is impossible because for $s<1$ the width of the distribution $\sigma^2\sim\ell^2/(1-s^2)$ is constant for large N and therefore it would collapse to zero in the limit $\ell\to 0$. The known solution for $s=1$ is useless, cannot be extrapolated, outside this precise value of s.

The first thing to notice when $s\in(1/2,1)$ is that there are no ''holes'' in the distribution when $N\to\infty$, that is, every point $x\in[-1/(1-s),1/(1-s)]$ can be reached by, at least, one path. In order to prove this, consider some arbitrary value $x\geqslant 0$ in the interval and let us construct a path ending there (the extension to $x\leqslant 0$ is trivial). Starting from the origin, let us make m steps to the right until we *exceed* the value of x. The distance Δ to the point x is smaller or equal to the last step $\Delta\leqslant s^{m-1}$. From there on, we continue the path but going to the *left* by r additional steps until we cross again the position x. If $s>1/2$, it is guaranteed that this second crossing occurs after a finite number of steps, because the maximal distance that we may go to the left, $s^m/(1-s)$, is larger than Δ. Clearly we are now closer to the location x. Now we start walking to the *right* until the next crossing. In the limit, we approach the point as much as we like. The completeness property of the real numbers completes the proof. One can prove that the number of steps between two changes of direction is *finite*. Therefore, since the total number of steps is infinite, the paths that we have just constructed will have an infinite number of turning points [except, of course, when x takes the extreme value $1/(1-s)$ that has only one path to reach it without any turning point].

Now we can prove that, under some condition, there is an *infinite* number of paths arriving at the same location, that is, the process is no longer retrodictive. For this, we consider that the path constructed above will have an infinite number of turning points where the direction of propagation, right or left, changes. We will show that at each change of direction we can choose another path, different from the one considered above. For this, we can make one step *away* from the location x, just *before* crossing it, and then change direction again toward x. Since this extra step has taken us away from x, the condition that guarantees that we will cross the location x becomes stronger. With a similar reasoning as above, we find that the *sufficient* condition is $s\geqslant 2/3$. Under this condition for the step factor, there are then infinite paths ending at any location x. In the region where this condition is not satisfied, that is, for $1/2<s<2/3$ no general statement could be found but one can find an infinite number of points such that each one of them can be reached by an infinite number of paths.

The existence of density function for the case $s=1/2$, $N\to\infty$ can be used to find the density for an infinite set of values of s. For example, let us consider the case $s=1/\sqrt{2}$. Separating the even and odd powers in Eq. (1), we have

$$X_I\left(\infty,\frac{1}{\sqrt{2}}\right)=\sum_{k\ \text{even}}^{\infty}\pm\left(\frac{1}{\sqrt{2}}\right)^k+\sum_{k\ \text{odd}}^{\infty}\pm\left(\frac{1}{\sqrt{2}}\right)^k. \quad (16)$$

Now, taking a common factor $1/\sqrt{2}$ from the odd powers, there remain two sums over even powers that can be renamed with another index running through *all* integers, leaving

$$X_I\left(\infty,\frac{1}{\sqrt{2}}\right)=\sum_{k=0}^{\infty}\pm\left(\frac{1}{2}\right)^k+\left(\frac{1}{\sqrt{2}}\right)\sum_{k=0}^{\infty}\pm\left(\frac{1}{2}\right)^k. \quad (17)$$

We see that the two sums correspond to the process for $s=1/2$ and therefore the random variable $X_I(\infty,1/\sqrt{2})$ is the sum of two random variables with known density function (uniform)

$$X_I\left(\infty,\frac{1}{\sqrt{2}}\right)=X_{I'}\left(\infty,\frac{1}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)X_{I''}\left(\infty,\frac{1}{2}\right). \quad (18)$$

The resulting density function is the convolution of two uniform density distributions. This result can be generalized for $s=1/\sqrt[n]{2}$ resulting in n convolutions of uniform density functions. When $n\to\infty$ these multiple convolutions approach the Gaussian distribution as is expected because $1/\sqrt[n]{2}\to 1$ for $n\to\infty$. These results are illustrated by a numerical simulation of the process taking $N=15$. This value is large enough because larger values of N do not result in significant changes in the distributions. In Figs. 2(a) and 2(b) we find histograms for the density distribution function for the cases $s=1/\sqrt{2}$ and $s=\sqrt[6]{2}$. In the first case, the distribution clearly results from the convolution of two uniform distributions and in the second case, the resulting distribution is very close to a Gaussian as the fit shows. These two smooth distributions are in contrast with the cases shown in Fig. 3 that clearly show overlapping self-similarity structure. Actually, the appearance of overlapping self-similarity is expected, as was explained, being a consequence of Eq. (6). Perhaps the interesting question is why the smooth cases do not show it explicitly. We can indeed see examples where the convolution of fractal distributions result in nonfractal smooth distributions. In order to clarify this, we can generalize the argument presented above with the conclusion that, if we know the density function for one value of $s=r$, then we can find the density function for an infinite number of values of $s=\sqrt[n]{r}$ as the n-fold convolution of the known distribution. Now let us take $r<1/2$ corresponding to a fractal distribution and choose n such that $\sqrt[n]{r}$ corresponds to a step factor with a smooth distribution such as $1/\sqrt[m]{2}$. Therefore we find the, a bit unexpected, situation that the convolution of fractal distributions lead to a smooth continuous distribution. The simplest example of this is that the convolution of two fractal distributions for $s=1/4$ results in the uniform distribution corresponding to $s=1/2$. We have therefore seen the possibility that, due to overlapping, the self-similarity patterns can be completely whipped out. However, for this to happen some sort of ''fine tuning'' of the step factor is required and

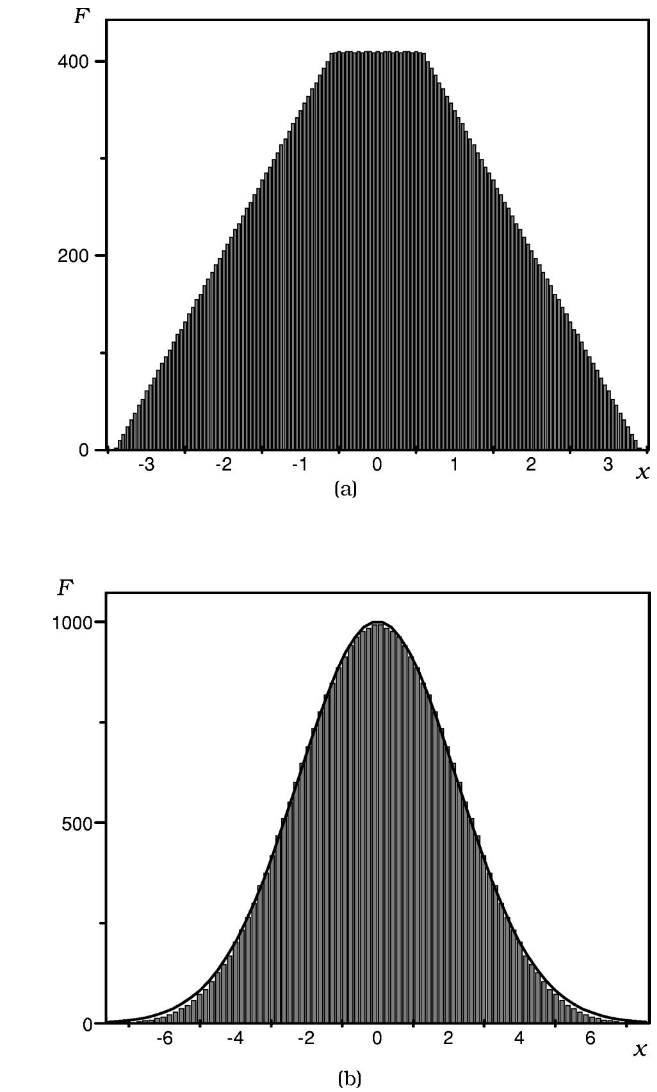

FIG. 2. Histograms for the number of paths F reaching an interval at the position x after $N=15$ steps with (a) $s=1/\sqrt{2}$ and (b) $s=1/\sqrt[6]{2}$. The first case is the convolution of two uniform distributions and the second case is very close to a Gaussian distribution as is shown by the fit.

therefore we expect that the set of values for the step factor with this property will have zero measure.

Playing with the space-time diagrams as in Fig. 1(a), we find that for $s<1/2$ the largest hole is centered around the position $X=0$ and has a width $(2-4s)/(1-s)$. When $1/2<s<1$, instead of a central hole we will have a central clustering of end points produced by the overlap of the set of paths that have the first step to the right with the set corresponding to the first step to the left. The width of the central cluster, that is, of the overlap region, is $(4s-2)/(1-s)$ (for large N). After the first step, the same reasoning is applied to explain the formation of another hole (if $s<1/2$) or cluster (if $1/2<s<1$) but now shifted to the right and to the left. So, an infinite number of holes or clusters are produced. In general we expect to find a cluster (or hole) at a distance $(1-s^n)/(1-s)$ from the center of width $s^n|4s-2|/(1-s)$ were $n=0,1,\ldots$. There is more in the case $1/2<s<1$ because each point within a cluster is the starting point of the same structure described. We will then have clusters within

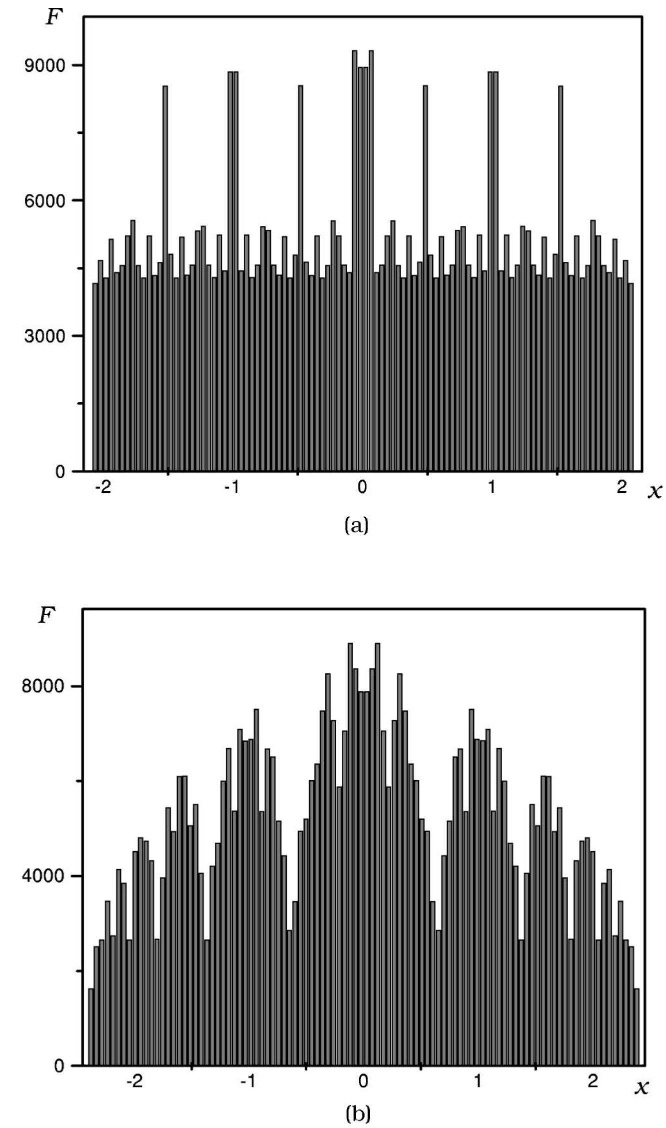

FIG. 3. Histograms for the number of paths F reaching an interval at the position x after $N=19$ steps with (a) $s=0.52$ and (b) $s=\sqrt{(\pi-1)/2\pi}\simeq 0.58$. Both cases show overlapping self-similarity. The width and location of the clusters agree with the values calculated.

clusters and so on. In Fig. 3, histograms for the end positions after $N=19$ steps with $s=0.52$ and $s=\sqrt{(\pi-1)/2\pi}\simeq 0.58$ are shown. This second value is of interest for the application of the process to a quantum-mechanical problem [8]. Notice that the first case is less than 5% away from the value $s=1/2$ where a smooth uniform density is generated, suggesting a very strong dependence of the distribution on the values of s. These histograms clearly show patterns of overlapping self-similarity explained above with the clusters having the predicted position and widths. In the case of Fig. 3(b), clusters within clusters are clearly seen.

We have seen that for most values of the step factor in the interval $[1/2,1]$, the density of the final positions will show overlapping self-similarity features. For values of s in a subset of zero measure, the corresponding density is a smooth function. We will now see that there is a set of values of s that have the property that different paths meet after a *finite* number of steps. Assume for instance, $s=(\sqrt{5}-1)/2$. In this case there are two different paths that return to the origin at the third step. They are $-1+s+s^2=0$ and $1-s-s^2=0$. These two paths form a square in the space-time diagram and we may therefore denote this case as a *square encounter*. There are *four* different paths that return to the origin after six steps. One of them is for instance $-1+s+s^2-s^3+s^4+s^5=-1+s+s^2+s^3(-1+s+s^2)=0$. Clearly, for this step factor and for $N\to\infty$, there is an infinite number of paths that return to the origin due to square encounters. Generalizing the argument, we find a countable infinite number of values for the step factor, such that the origin can be reached by an infinite number of paths caused by square encounters; because for each integer $m>2$ there is a value of s such that the path returns to the origin at the mth step; with the first step to the right and all other steps to the left (or the opposite). The value of the step factor is the root of $s^m-2s+1=0$ [there is always a real root in the interval $(1/2,1)$]. In the example above we have $m=3$. These encounters correspond to the case of vanishing gap $G_m=0$ of the drifting picture with $s<2$. This property of the origin is not unique because, as already mentioned, every point visited can be considered to be the starting point of the (scaled) process.

Another type of encounter after finite number of steps is when different paths starting at the origin meet again at the mth step, but not at the origin. Due to the left-right symmetry, there is another meeting point at the same coordinate but with reversed sign. Each one of these two meeting points is the starting point for the same process (with a common factor s^m) resulting in four meeting points. Clearly, in the limit, there will be an infinite number of points with this property. The value of the step factor that produces such a behavior in the simplest case is the solution of $+1-s-s^2-s^3=-1-s+s^2+s^3$, that is, $s^3+s^2-1=0$ with a root $s\approx 0.755$. This last example can be generalized to a class of cases that we may call *rectangular encounters* because the two paths form a rectangle in a space-time diagram. Imagine one path making the first step to the right and all other steps to the left. We can find a value of s such that this path will encounter a path made of the first n steps to the left followed by r steps to the right. For this encounter, the step factor must be such that $1=s^n+s^{n+1}+\dots+s^{n+r-1}$, that is, $(1-s^r)/(1-s)=1/s^n$. For every pair of integers $n\geqslant 2, r\geqslant 2$ there is a solution in the interval $s\in(1/2,1)$. The example above corresponds to $n=r=2$. Aside from the square and the rectangular encounters we can expect more complicated encounters corresponding to closed polygons in the space-time diagrams. In addition to these *polygonal encounters*, there may be encounters of paths having a different number of steps. In the space-time diagram, they would result in an open figure suggesting the name of *open encounters*. The general treatment of all possible encounters seems to be a difficult problem, equivalent to the study of all possible polynomials with coefficients $0, \pm 1, \pm 2$ with root in $[1/2,1]$. It has been proven [11] that the set of values of s corresponding to these encounters has zero measure and there is an open mathematical conjecture [13] that the set is at most countable.

The remarkable thing about the encounters is that, in these cases, the distribution becomes singular and does not have a density function. The intuitive reason for this is that, when we have encounters, there are locations with a strong accumulation of paths reaching it. A distribution will have a density function if the number of paths falling within an interval, divided by the total number of paths (2^N) and by the size of the interval, goes to a constant when the size of the interval vanishes. This constant is proportional to the value of the density function at the point. We can see that this is not true for every interval in the case that we have encounters [12]. Let us take for instance a step factor producing square encounters after three steps (the simplest one). Consider N steps multiple of 3, $N=3n$, and an interval of measure s^{3n} around the center $x=0$ [that is, equal to the length of the $(N+1)$th step]. The number of paths falling in this interval due to the square encounters is 2^n. There are actually more paths falling in the interval aside from those with encounters. Therefore R, the relative number of paths falling in the interval per unit interval length, is such that

$$R\geqslant \frac{2^n}{2^{3n}s^{3n}}=\left(\frac{1}{4s^3}\right)^n\approx(1.06)^n, \tag{19}$$

where we have used the value of the step factor for this square encounter $s=(\sqrt{5}-1)/2$. Clearly, in the limit where the interval vanishes, $n\to\infty$, the ratio R diverges indicating that there is no density function.

IV. SUMMARY AND CONCLUSIONS

The random walk with exponentially varying step has been presented from a physical perspective describing damped $(s<1)$ or amplified $(s>1)$ diffusion. $s=1$ is therefore like a critical value separating two phases of bound or unbounded diffusion. The symmetry under the transformation $s\to 1/s$ for finite number of steps N relates these two phases and reduces the range of study to the values $s\leqslant 1$. The dependence of the process on the value of the step factor is summarized in Fig. 1(b). When $s<1/2$ or $s>2$, the process is retrodictive and the set of final positions is a fractal. For step factors in the interval $s\in[1/2,2]$, when $N\to\infty$, the process is no longer retrodictive but, in most cases, overlapping self-similarity patterns appear. For some special values of the step factor, the density becomes smooth. Another set of values of the step factor, of zero measure, generates a different type of encounters of the paths after finite number of steps. These encounters result in singular distributions without density function. A complete study of this region is however, an unsolved problem that has been treated from the mathematical perspective of random series as a special case of Bernoulli convolutions [11,13,14] with emphasis on the mathematical generalizations. The aim of this paper is to present the properties of the process in a simple and attractive way with emphasis on the physical aspects of the problem. We hope with this contribution to motivate further progress on this subject.

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